Pseudo-hermitian interaction between an oscillator and a spin-$\frac{1}{2}$ particle in the external magnetic field

Bhabani Prasad Mandal *

Department of Physics, Banaras Hindu University,
Varanasi-221005, India

Abstract

We consider a spin-$\frac{1}{2}$ particle in the external magnetic field which couples to a harmonic oscillator through some pseudo-hermitian interaction. We find that the energy eigenvalues for this system are real even though the interaction is not PT invariant.

1 Introduction

In the last few years the study of some nonhermitian Hamiltonian with real spectrum have given rise to a growing interest in the literature. This was mainly initiated by Bender and Boettcher's observation that with properly defined boundary conditions the spectrum of the Hamiltonian $H = p^2 + x^2(ix)^\nu, \quad (\nu \geq 0)$ is real, positive and discrete. The reality of the spectrum is a consequence of unbroken PT [combined parity (P) and time reversal (T)] invariance of the Hamiltonian i.e. $[H, PT] = 0$ [1, 2]. However pairs of complex conjugate eigenvalues appear when the PT symmetry is broken spontaneously. This is also illustrated nicely with the help of a nonhermitian but PT invariant potential with quasi-exactly solvable eigenvalues [3] .

This surprising result attracts lot of interest in last few years and many other such nonhermitian but PT symmetric systems, mostly for one particle in one space dimension have been investigated [4]- [17]. Validity of these results have also been tested for the cases of nonhermitian extension of some exactly solvable many particle quantum systems in one dimension [18] -[22]. Nonhermitian extension of some field theoretic models has been considered in Refs. [5, 6].

However to develop a consistent quantum theory for these nonhermitian Hamiltonians one encounters serve difficulties [10, 13]. Firstly, the eigenstates of PT symmetric nonhermitian Hamiltonians with real eigenvalues only do not satisfy standard completeness relations. More importantly if one takes the natural inner product associated with PT-symmetric system as

$$(f, g) = \int d^4x[PTf(x)]g(x),$$

*e-mail address: bpm@bose.res.in

then the half of the energy eigenstates have negative norms which makes it difficult to maintain the familiar probabilistic interpretation of quantum theory. Recently Bender and coworkers have found a new symmetry , C, inherent in all such Hamiltonian with unbroken PT symmetry[2, 4]. This allows to introduce an inner product structure associated with CPT conjugation for which the norms of the quantum states are positive definite and one gets usual completeness relation. As a result the Hamiltonian and its eigenstates can be extended to complex domain so that the associated eigenvalues are real and underlying dynamics is unitary.

In an another approach Mostafazadeh [7, 8] has shown that the reality of spectrum of nonhermitian Hamiltonian is due to so called pseudo-hermiticity properties of the Hamiltonian. A Hamiltonian is called η pseudo-hermitian if it satisfies the relation

$$\eta H \eta^{-1} = H^{\dagger}, \tag{1.1}$$

where η is a linear hermitian operator. All PT symmetric nonhermitian Hamiltonian are pseudo-hermitian and these consist a subclass of pseudo-hermitian Hamiltonian. All the observations of PT symmetric nonhermitian Hamiltonian can be explained nicely in this approach.

The purpose of this letter is to consider an example of nonhermitian Hamiltonian which is not PT invariant but pseudo-hermitian and study the different properties of such system. With this aim we consider a system consisting of a spin half particle in the external magnetic field coupled to an oscillator via nonhermitian interaction. We find that the spectrum is real even though the interaction term is not PT symmetric. The explicit PT asymmetric system has also been considered recently [23].

Here is the plan of the paper. In section II we will discuss the Hamiltonian of the system and its symmetries. We will find the energy eigenvalues and corresponding eigenfuctions explicitly for this system in section III. Section IV is kept for concluding remarks.

2 The Model

We consider a system of a spin $\frac{1}{2}$ particle in the external magnetic field, $\vec{B}$ coupled to an oscillator through some nonhermitian interaction described by the Hamiltonian

$$H = \mu \vec{\sigma} \cdot \vec{B} + \hbar \omega a^{\dagger} a + \rho(\sigma_{+} a - \sigma_{-} a^{\dagger}). \tag{2.1}$$

Here $\vec{\sigma}$'s are Pauli matrices, ρ is some arbitrary real parameter and $\sigma_{\pm} \equiv \frac{1}{2}[\sigma_x \pm i\sigma_y]$ are spin projection operators. $a, a^{\dagger}$ are usual creation and annihilation operator for the oscillator states and defined as

$$a = \frac{p - im\omega x}{\sqrt{2m\omega\hbar}}, \quad a^{\dagger} = \frac{p + im\omega x}{\sqrt{2m\omega\hbar}}, \tag{2.2}$$

with

$$a|n> = \sqrt{n}|n-1>, \quad \text{and} \quad a^\dagger|n> = \sqrt{n+1}|n+1>, \tag{2.3}$$

where the notation $|n>$ for number eigenvectors for the oscillator has been adopted.

For the sake of simplicity we can choose the external magnetic field in z-direction, $\vec{B} = B_0\hat{z}$ and the Hamiltonian for the system as given in Eq.(2.1) is reduced to ,

$$H = \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a + \rho(\sigma_+ a - \sigma_- a^\dagger), \tag{2.4}$$

where $\epsilon = 2\mu B_0$. This system can also be thought of a two level system coupled to an oscillator where ϵ is the splitting between the levels. Note that this Hamiltinian is not hermitian as,

$$\begin{aligned} H^\dagger &= \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a - \rho(\sigma_+ a - \sigma_- a^\dagger), \\ &\neq H, \end{aligned} \tag{2.5}$$

as $\sigma_\pm^\dagger = \sigma_\mp$. Under parity transformation [i.e. $x \longrightarrow -x$; $p \longrightarrow -p$] both $\vec{\sigma}$ and $\vec{B}$ do not change sign as both are axial vectors but as it clear from the Eq (2.2) that both the creation and annihilation operators change sign.

$$\begin{aligned} &P\vec{\sigma}P^{-1} = \vec{\sigma}, \\ &P\vec{B}P^{-1} = \vec{B}, \\ &PaP^{-1} = -a, \\ &Pa^\dagger P^{-1} = -a^\dagger. \end{aligned} \tag{2.6}$$

Note the interaction term of the Hamiltonian in Eq. (2.1) changes sign under parity operation. The time reversal operator for the system of spin half particles is $T = -i\sigma_y K$ where K is complex conjugation operator. We note the changes of following quantities under time reversal transformation as,

$$\begin{aligned} &T\vec{\sigma}T^{-1} = -\vec{\sigma}, \\ &T\vec{B}T^{-1} = \vec{B}, \\ &T\sigma_\pm T^{-1} = -\sigma_\mp, \\ &TaT^{-1} = -a, \\ &Ta^\dagger T^{-1} = -a^\dagger. \end{aligned} \tag{2.7}$$

We have considered the magnetic field as the external element in our system and it does not change sign under time reversal operation. However one can consider magnetic field in other way also, when it changes sign under time reversal as the current producing magnetic field is reversed under time reversal. The results of this paper are same in both cases. From Eqs. (2.6) and (2.7) we can see that the Hamiltonia in Eq. (2.1) is not PT symmetric,

$$\begin{aligned} PT\ H\ (PT)^{-1} = &- \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a + \rho(\sigma_+ a^\dagger - \sigma_- a), \\ &\neq H. \end{aligned} \tag{2.8}$$

However this Hamiltonian is σ_z-pseudo-hermitian

$$\begin{aligned}\sigma_z H \sigma_z^{-1} &= \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a + \rho(\sigma_z\sigma_+\sigma_z a - \sigma_z\sigma_-\sigma_z a^\dagger), \\ &= \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a - \rho(\sigma_+ a - \sigma_- a^\dagger), \\ &= H^\dagger. \end{aligned} \tag{2.9}$$

Here we have used the relations $\sigma_z\sigma_\pm\sigma_z = -\sigma_\pm$. In case of $\eta-$ pseudo-hermitian Hamiltonian the choice of the operator η is not unique [7]. Therefore we look for whether our Hamiltonian is pseudo-hermitian with respect to any other operator. Indeed it is also pseudo-hermitian with respect to parity operator as,

$$\begin{aligned}PHP^{-1} &= \frac{\epsilon}{2}P\sigma_z P^{-1} + \hbar\omega P a^\dagger a P^{-1} + \rho(P\sigma_+ a P^{-1} - P\sigma_- a^\dagger P^{-1}), \\ &= \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a - \rho(\sigma_+ a - \sigma_- a^\dagger), \\ &= H^\dagger. \end{aligned} \tag{2.10}$$

Finally we found a symmetry of our Hamiltonian . It is invariant under the symmetry generated by the combined operator, $P\sigma_z$ i.e.

$$[H, P\sigma_z] = 0. \tag{2.11}$$

However it not surprising as it is shown in Ref.[7] that if a Hamiltonian is pseudo-hermitian with respect to two different operator η_1, η_2 then the system is symmetric under the transformation generated by $\eta_2^{-1}\eta_1$.

3 The solutions

To find the energy eigenvalues and corresponding eigenvectors of the system described by the Hamiltonian in the Eq. (2.4) we adopt the notation for the state as, $|n, \frac{1}{2}m_s>$ where n is eigenvalue for the number operator $a^\dagger a$ i.e. $a^\dagger a|n>= n|n>$ and $m_s = \pm 1$ are the eigenvalues of the operator σ_z i.e. $\sigma_z|\frac{1}{2}m_s>= m_s|\frac{1}{2}m_s>$. It is readily seen that $|0, -\frac{1}{2}>$ is a ground state of the Hamiltonian with eigenvalue $-\frac{\epsilon}{2}$ and it is non-degenerate.

$$H|0, -\frac{1}{2}>= -\frac{\epsilon}{2}|0, -\frac{1}{2}> . \tag{3.1}$$

Note that the projection operators $\sigma_\pm$ have the following usual properties when they act on the state $|n, \pm\frac{1}{2}>$,

$$\begin{aligned} \sigma_+|n, \frac{1}{2}>= 0; \qquad & \sigma_+|n, -\frac{1}{2}>= |n, \frac{1}{2}>, \\ \sigma_-|n, -\frac{1}{2}>= 0; \qquad & \sigma_-|n, \frac{1}{2}>= |n, -\frac{1}{2}> . \end{aligned} \tag{3.2}$$

We observe that the next possible states $|0, \frac{1}{2}>$ is not a eigenstate of the Hamiltonian,

$$H|0, \frac{1}{2}> = \frac{\epsilon}{2}|0, \frac{1}{2}> - \rho|1 - \frac{1}{2}> . \tag{3.3}$$

However this state along with the state $|1, -\frac{1}{2}>$ close under the action of the Hamiltonian and form a invariant subspace in the space of states as,

$$H|1, -\frac{1}{2}> = (\hbar\omega - \frac{\epsilon}{2})|1, -\frac{1}{2}> + \rho|0, \frac{1}{2}> . \tag{3.4}$$

First two excited states belong to this sector spanned by these two states, $|0, \frac{1}{2}>$ and $|1, -\frac{1}{2}>$ wherein the Hamiltonian matrix is given by†

$$H_1 = \begin{bmatrix} \frac{\epsilon}{2} & \rho \\ -\rho & -\frac{\epsilon}{2} + \hbar\omega \end{bmatrix}.$$

The eigenvalues of this Hamiltonian matrix are given by $\lambda_1^{I,II} = \frac{1}{2}\left[\hbar\omega \pm \sqrt{(\hbar\omega - \epsilon)^2 - 4\rho^2}\right]$. Note these eigenvalues are real provided $|\hbar\omega - \epsilon| \geq 2\rho$. Putting $2\rho = (\hbar\omega - \epsilon)\sin\theta_1$ we find the eigenvectors corresponding to this doublet are

$$\begin{aligned} |\Psi_1^I> &= \cos\frac{\theta_1}{2}|0, \frac{1}{2}> + \sin\frac{\theta_1}{2}|1, -\frac{1}{2}>, \quad \text{for } \lambda_1^I = \frac{\hbar\omega}{2}(1 + \cos\theta_1) - \frac{\epsilon}{2}\cos\theta_1, \\ |\Psi_1^{II}> &= \sin\frac{\theta_1}{2}|0, \frac{1}{2}> + \cos\frac{\theta_1}{2}|1, -\frac{1}{2}>, \quad \text{for } \lambda_2^{II} = \frac{\hbar\omega}{2}(1 - \cos\theta_1) + \frac{\epsilon}{2}\cos\theta_1. \end{aligned} \tag{3.5}$$

It may be observed that these two states are not orthogonal to each other nor do they have to be as $H \neq H^\dagger$. To find the next excited states we have to consider next invariant subspace. It can be easily checked that next invariant subspace is spanned by the vectors,$|1, \frac{1}{2}>, |2, -\frac{1}{2}>$ and the eigenvalues and eigenvectors for this doublet can be obtained following the same method.

The result is easily generalized to the sector spanned by $|n, \frac{1}{2}>$ and $|n + 1, -\frac{1}{2}>$ wherein the Hamiltonian matrix is given by,

$$H_{n+1} = \begin{bmatrix} \frac{\epsilon}{2} + n\hbar\omega & \rho\sqrt{n+1} \\ -\rho\sqrt{n+1} & -\frac{\epsilon}{2} + (n+1)\hbar\omega \end{bmatrix}.$$

Now we have the eigenvalues of this Hamiltonian matrix are given by

$$\lambda_{n+1}^{I,II} = \frac{1}{2}\left[(2n+1)\hbar\omega \pm \sqrt{(\hbar\omega - \epsilon)^2 - 4\rho^2(n+1)}\right]. \tag{3.6}$$

†Similar two by two matrix Hamiltonian is also considered in ref. [24] for a completely different system.

These eigenvalues are real provided $|\hbar\omega - \epsilon| \geq 2\rho\sqrt{n+1}$. Now putting $2\rho\sqrt{n+1} = (\hbar\omega - \epsilon)\sin\theta_{n+1}$, we find the eigenvectors corresponding to this doublet are

$$\begin{aligned}
|\Psi^{I}_{n+1}> &= \cos\frac{\theta_{n+1}}{2}|n,\frac{1}{2}> + \sin\frac{\theta_{n+1}}{2}|n+1,-\frac{1}{2}>, \\
&\quad \text{for } \lambda^{I}_{n+1} = \frac{\hbar\omega}{2}(2n+1+\cos\theta_{n+1}) - \frac{\epsilon}{2}\cos\theta_{n+1}, \\
|\Psi^{II}_{n+1}> &= \sin\frac{\theta_{n+1}}{2}|n,\frac{1}{2}> + \cos\frac{\theta_{n+1}}{2}|n+1,-\frac{1}{2}>, \\
&\quad \text{for } \lambda^{II}_{n+1} = \frac{\hbar\omega}{2}(2n+1-\cos\theta_{n+1}) + \frac{\epsilon}{2}\cos\theta_{n+1}.
\end{aligned} \tag{3.7}$$

We observe that these eigenstates are also eigenstate of the operator, $P\sigma_z$

$$P\sigma_z|\Psi^{I,II}_{n+1}>=(-1)^n|\Psi^{I,II}_{n+1}>, \tag{3.8}$$

as $P|n,\pm\frac{1}{2}>=(-1)^n|n,\pm\frac{1}{2}>$ and $\sigma_z|n,\pm\frac{1}{2}>=\pm|n,\pm\frac{1}{2}>$. Thus we have real eigenvalues when the symmetry is not broken. In the regime $|\hbar\omega-\epsilon| < 2\rho\sqrt{n+1}$ the eigenvalues for a particular doublet become complex conjugate. For $n+1$ th doublet, the complex eigenvalues are $\frac{1}{2}\left[\hbar\omega(2n+1) + \frac{i}{2}\sqrt{4\rho^2(n+1)-(\hbar\omega-\epsilon)^2}\right]$.

The Hamiltonian of the system has a complete set of biorthonormal eigenvectors. To see this we find the eigenvalues and eigenvectors of $H^\dagger$ as,

$$\begin{aligned}
|\Phi^{I}_{n+1}> &= \cos\frac{\theta_{n+1}}{2}|n,\frac{1}{2}> - \sin\frac{\theta_{n+1}}{2}|n+1,-\frac{1}{2}>, \\
&\quad \text{for } \tilde{\lambda}^{I}_{n+1} = \frac{\hbar\omega}{2}(2n+1+\cos\theta_{n+1}) - \frac{\epsilon}{2}\cos\theta_{n+1}, \\
|\Phi^{II}_{n+1}> &= -\sin\frac{\theta_{n+1}}{2}|n,\frac{1}{2}> + \cos\frac{\theta_{n+1}}{2}|n+1,-\frac{1}{2}>, \\
&\quad \text{for } \tilde{\lambda}^{2}_{n+1} = \frac{\hbar\omega}{2}(2n+1-\cos\theta_{n+1}) + \frac{\epsilon}{2}\cos\theta_{n+1}.
\end{aligned} \tag{3.9}$$

From the Eqs. (3.7) and (3.9) one can easily check that

$$< \Psi^i_n|\Phi^j_m >= \delta_{nm}\delta_{ij} \quad \text{where } i,j = I \text{ or } II, \tag{3.10}$$
$$\sum_n\sum_i |\Psi^i_n >< \Phi^i_n| = \sum_n\sum_i |\Phi^i_n >< \Psi^i_n| = \mathbf{1}, \tag{3.11}$$

modulo a constant scaling of the states.

Now we observe that one state in each doublet has $P\sigma_z$ pseudo norm negative

$$< \Psi^i_n|P\sigma_z|\Psi^i_m >= \pm(-1)^n\delta_{mn} \text{ for } i = I, II. \tag{3.12}$$

This fact is similar to the fact in all PT-symmetric nonhermitian systems where the PT-norms for half of the states are negative. The system we have considered is invariant under the the symmetry generated by the combined operator $P\sigma_Z$ and we have $P\sigma_z$ norms for

half of the states are negative. Now following the work in ref [2] we introduce the extra symmetry, C, connected with the equal numbers of positive and negative norms. The operator C is the observable that represents the measurement of signature of $P\sigma_z$ norm of a state. Then $PC\sigma_z$ norms for all the states are positive and definite. Thus if we introduce the inner product associated with our system as,

$$(f, g) = \int d^4x[P\sigma_z Cf(x)]g(x), \tag{3.13}$$

then all the states become orthonormal.

4 Conclusion

We consider a system of a spin half particle in external magnetic field coupled to an oscillator through nonhermitian interaction. The Hamiltonian for this system is not PT-symmetric but it is pseudo-hermitian with respect to two different operators, P and σ_z and hence symmetric under the transformations generated by the operator $P\sigma_z$. We found that, except the ground state, all other states occurs in doublet due to the interaction of oscillator with two levels system. The energy eigenvalues corresponding to all these states are real when the symmetry $P\sigma_z$ is unbroken. We have also shown that the Hamiltonian of the system has a complete set of biorthonormal eigenvectors. However with the conventional definition of scalar product the eigenstates corresponding to a particular doublet are not orthogonal to each other, but they are orthogonal to all other eigenstates corresponding to other doublets. The eigenstates of a particular doublet are orthogonal to each other only when, $\theta_n = n\pi$, i.e. $\rho = 0$. That is the situation when the nonhermitian interaction drops. However when the $P\sigma_z$ symmetry is broken spontaneously the energy eigenvalues are complex and all the eigenstates are orthogonal to each other. This implies that when energy eigenvalues are observable(real), it is not possible to have all the states orthogonal to each other with the conventional definition of scalar product, on the other hand all the eigenstates satisfy orthonormality condition when eigenvalues (complex) are not observable. This result is not surprising as the system is nonhermitian [10]. However all the states satisfy orthonormality with respect to the new definition of inner product in Eq. 3.13.

References

[1] C.M. Bender and S. Boettcher, *Phys. Rev. Lett.* **80** (1998) 5243; *J. Phys.* **A 31** (1998) L273; C.M. Bender, S. Boettcher and P.N. Meisinger, *J. Math. Phys.* **40** (1999) 2210.

[2] C.M. Bender and S. Boettcher, *Phys. Rev. Lett.* **89** (2002) 270401-1;

[3] A. Khare and B. P. Mandal, *Phys. Lett.* **A 272** (2000) 53.

[4] C.M. Bender,J. Brod, A. Refig and M. Reuter, quant-ph/0402026.

[5] C.M. Bender, D.C. Brody and H. F. Jones, hep-th/0402183; hep-th/0402011.

[6] K. A. Milton, *Czech J. Phys.* **54** (2004) 1069.

[7] A. Mostafazadeh, *J. Math Phys.* **43** (2002) 205; **43** (2002) 2814; **43** (2002) 3944.

[8] A. Mostafazadeh, *Nucl. Phys. B* **640** (2002) 419; *J. Math. Phys.* **44** (2003) 974; quant-ph/0307059; quant-ph/0404025; quant-ph/0304080.

[9] S. Weigert, quant-ph/0209054 and quant-ph/0306040.

[10] G. Japaridze, *J. Phys. A* **35** (2002), 1709.

[11] Z. Ahmed, *Phys. Lett. A* **294** (2002), 287.

[12] M. Znojil, math-ph/0104012, quant-ph/0303122 & math-ph/0403033.

[13] P. Dorey, C. Dunning and R. Tateo, *J. Phys.* **A 34** (2001) 5679.

[14] B.Bagchi, C. Quesne, *Phys. Lett.* **A273** (2000) 256.

[15] S. Rajvjani, A.K. kapoor and P. K. Panigrahi, quant-ph/0403054.

[16] A. Sinha, G. Levai and P. Roy quant-ph/0401064.

[17] A. Khare and U. Sukhatme, quant-ph/0402106

[18] B. Basu-Mallick and B.P. Mandal, *Phys. Lett.* **A 284** (2001) 231.

[19] B. Basu-Mallick, *Int. J. of Mod. Phys. B*, **16** (2002) 1875.

[20] B. Basu-Mallick, T. Bhattacharyya A. Kundu, and B. P. Mandal *Czech. J. Phys* **54** (2004) 5.

[21] B. Basu-Mallick, T. Bhattacharyya and B. P. Mandal, nlin.SI/0405068 [To appear in IJMPA, 2004].

[22] Y. Brihaye and A. Ninimahazwe hep-th/0311081

[23] E. Caliceti, F. Cannata, M. Znojil & A. Ventura math-ph/0406031 .

[24] A. Mostafazadeh, quant-ph/0310164.